THE PROPAGATION OF THERMAL STRESSES IN AN ELASTIC-PLASTIC BAR

(RASPROSTRANENIE TEMPERATURNYKH NAPRIAZHENII V UPRUGO-PLASTICHESKOM STERZHNE)

PMM Vol.27, No.2, 1963, pp. 383-389

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(Received September 17, 1962)

This paper treats the problem of the propagation of stress waves in a semi-infinite elastic-plastic bar resulting from sudden heating of the free end; moreover, account is taken of the dependence of the coefficient of thermal conductivity on the temperature (i.e. the nonlinear equation of heat conduction will be used), which [1,2] gives rise to a finite velocity of propagation of heat in the bar. The consideration of this fact leads to quantitative features of the elastic solution differing from well-known earlier solutions (e.g. [3]). It will be assumed that the material of the bar is incompressible and linearly elastic; the mechanical properties will be regarded as being independent of the temperature.

1. The solution of the problem leads to the solution of the following system of equations

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial \sigma}{\partial x} \qquad \text{(equation of motion)} \tag{1.1}$$

$$\sigma = E\left(\frac{\partial v}{\partial x} - \alpha \vartheta\right) \qquad \text{(equation of state)} \tag{1.2}$$

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial \vartheta}{\partial x}\right) \qquad \text{(equation of heat conduction)} \tag{1.3}$$

where σ is the stress, v the displacement, ρ the density, ϑ the temperature, α the coefficient of thermal expansion and λ the coefficient of thermal conductivity. We will assume a power-law dependence of λ on ϑ of the following form

$$\lambda = \lambda_0 \frac{n\alpha^{n-1}}{e_s^{\,n-1}} \vartheta^{n-1} \qquad (n > 1) \qquad \left(e_s = \frac{\sigma_s}{E}\right)$$

where σ_s is the yield stress. Then, equation (1.3) can be written as

$$\frac{\partial \vartheta}{\partial t} = \lambda_0 \frac{\alpha^{n-1}}{e_s^{\,n-1}} \frac{\partial^2 \vartheta^n}{\partial x^2}$$

In the system of dimensionless variables

$$y = \frac{\sqrt{E}}{\lambda_0 \sqrt{\rho}} x, \quad \tau = \frac{E}{\lambda_0 \rho} t, \quad T = \frac{\alpha \vartheta}{e_s}, \quad u = \frac{\sqrt{E}}{\lambda_0 \sqrt{\rho} e_s} v, \quad s = \frac{\sigma}{\sigma_s} \tag{1.4}$$

the initial system of equations assumes the form

$$\frac{\partial^2 u}{\partial \tau^2} = \frac{\partial s}{\partial y}, \qquad s = \frac{\partial u}{\partial y} - T, \qquad \frac{\partial T}{\partial \tau} = \frac{\partial^2 T^m}{\partial y^2} \tag{1.5}$$

The initial and boundary conditions will be

$$u(y, 0) = 0, \qquad \partial u(y, 0) / \partial \tau = 0, \qquad s(0, \tau) = 0 \tag{1.6}$$

$$T(y, 0) = 0, \qquad T(0, \tau) = T_0 \tag{1.7}$$

For definiteness, we will set $n = 2$ in (1.5). We solve the resulting equations by means of the method of Kàrmàn-Pohlhausen; the solution will be sought in the form

$$T = a_0 + a_1 \frac{y}{l(\tau)} + a_2 \left[\frac{y}{l(\tau)}\right]^2 + \ldots \tag{1.8}$$

where $T \equiv 0$ for $y < l(\tau)$, $l(\tau) \to 0$ as $\tau \to 0$. In order to simplify the resulting calculations, consideration will be restricted to the first two terms in the expansion (1.8). It can be verified directly, that the calculation of a large number of terms does not change the qualitative picture of the solution for the stresses. Taking condition (1.7) into account, we obtain

$$T - \begin{cases} T_0 \left[1 - y / l(\tau)\right], & y \leqslant l(\tau) \\ 0. & y \geqslant l(\tau) \end{cases} \tag{1.9}$$

Satisfying the last of equations (1.5) in the mean with respect to y, we have for $l(\tau)$ the equation

$$\int_0^{l(\tau)} \frac{\partial T}{\partial \tau} dy = \int_0^{l(\tau)} \frac{\partial^2 T^2}{\partial y^2} dy$$

Now substituting expression (1.9) for T and carrying out the integrations, we obtain

$$\frac{1}{2}\frac{dl}{d\tau}=\frac{2T_0}{l(\tau)}, \qquad l(0)=0 \tag{1.10}$$

Hence

$$l(\tau)=\beta\sqrt{\tau}\,(\beta=2\sqrt{2T_0})$$

2. From (1.5) we obtain the equation of motion in terms of the displacements

$$\frac{\partial^2 u}{\partial\tau^2}=\frac{\partial^2 u}{\partial y^2}-Q(y,\tau), \qquad Q=\frac{\partial T}{\partial y} \tag{2.1}$$

The solution of this equation in the region bounded by the Oy-axis and the bisector $\tau = y$ (Fig. 1) for the initial conditions (1.8) can be given by d'Alembert's formula

$$u(B)=-\frac{1}{2}\iint\limits_{\Delta ABC} Q(\xi,\eta)\,d\xi\,d\eta=-\frac{1}{2}\int_0^\tau d\eta\int_{y+\eta-\tau}^{y-\eta+\tau} Q\,d\xi=$$

$$=-\frac{1}{2}\int_0^\tau [T(y-\eta+\tau,\eta)-T(y+\eta-\tau,\eta)]\,d\eta \tag{2.2}$$

Taking (1.9) into account, we obtain the solution in the following form:

in the region $yOGN$

$$u=0$$

in the region $OGHO$

$$u=\frac{4T_0}{3\beta}\tau\sqrt{\tau}+T_0y-\frac{T_0\beta}{2}\left(\sqrt{{}^1\!/_4\beta^2+y+\tau}-\sqrt{{}^1\!/_4\beta^2-y+\tau}\right)-$$

$$-\frac{T_0}{\beta}(\tau-y)A^{1/2}(\tau-y)-\frac{T_0}{\beta}(\tau+y)A^{1/2}(\tau+y)+\frac{T_0}{3\beta}[A^{3/2}(\tau-y)+A^{3/2}(\tau+y)]$$

in the region $MHGN$

$$u=T_0\beta\sqrt{{}^1\!/_4\beta^2-y+\tau}-\frac{T_0}{\beta}(\tau-y)[A^{1/2}(\tau-y)-B^{1/2}(\tau-y)]+$$

$$+\frac{T_0}{3\beta}[A^{3/2}(\tau-y)-B^{3/2}(\tau-y)]$$

where

$$A(x)={}^1\!/_2\beta^2+x-\beta\sqrt{{}^1\!/_4\beta^2+x}, \qquad B(x)={}^1\!/_2\beta^2+x+\beta\sqrt{{}^1\!/_4\beta^2+x} \tag{2.3}$$

For the stresses, by formula (1.6), we obtain the following results:

in the region $yOGN$

$$s=0$$

in the region $OGHO$

$$s = \frac{T_0}{\beta}\left[A^{1/2}(\tau - y) - A^{1/2}(\tau + y)\right] - T_0\left(1 - \frac{y}{\beta\sqrt{\tau}}\right) \qquad (2.4)$$

in the region $MHGN$

$$s = \frac{T_0}{\beta}\left[A^{1/2}(\tau - y) - B^{1/2}(\tau - y)\right] \qquad (2.5)$$

On the bisector $\tau = \gamma$ we will have

$$s = -T_0 + \frac{T_0}{\beta}\sqrt{y} - \frac{T_0}{\beta}A^{1/2}(2y) \quad (y \leqslant \beta^2), \qquad s = -T_0 \quad (y \geqslant \beta^2) \qquad (2.6)$$

Hence it is clear that those lines in the region $MHGN$ which are parallel to the principal bisector will be lines of constant displacements, velocities, and stresses.

For the determination of the solution in the region τOM, we have the Cauchy problem:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \tau^2} + Q(y, \tau), \qquad u(0, \tau) = \varphi(\tau), \qquad u_y(0, \tau) = T_0 \qquad (2.7)$$

where $\varphi(\tau)$ is a yet unknown function; the indices y and τ indicate derivatives. By the d'Alembert formula we obtain (Fig. 1)

$$u(P) = \frac{1}{2}\varphi(\tau + y) + \frac{1}{2}\varphi(\tau - y) + T_0 y + \frac{1}{2}\iint\limits_{\Delta DPE} Q(\xi, \eta)\, d\xi\, d\eta \qquad (2.8)$$

From the condition of continuity of displacement on the bisector $\tau = y$ we find

$$\varphi(\gamma) = -T_0\gamma - \int_0^{\gamma} d\eta \int_0^{\gamma - \eta} Q(\xi, \eta)\, d\xi$$

Substituting this expression into (2.8), we obtain

$$u(y, \tau) = T_0(y - \tau) + \frac{1}{2}\iint\limits_{\Delta DPE} Q\, d\xi\, d\eta - \frac{1}{2}\iint\limits_{\Delta ODC} Q\, d\xi\, d\eta - \frac{1}{2}\iint\limits_{\Delta OEK} Q\, d\xi\, d\eta$$

Since $Q = \partial T/\partial y$ is independent of y, then, considering the function $Q(y, \tau)$ continued as an even function into the region of negative values of y, the solution can be rewritten in the form

$$u(y, \tau) = T_0(y - \tau) - \frac{1}{2}\iint\limits_{\Delta FPC} Q\, d\xi\, d\eta = T_0(y - \tau) - \frac{1}{2}\int_0^{\tau}\left[T(y + \tau - \eta, \eta) - T(y - \tau + \eta, \eta)\right] d\eta$$

By formula (1.5) for the stresses, taking account of (1.9), we obtain

in the region τOHL

$$s = \frac{T_0}{\beta}\,[A^{1/2}(\tau - y) - A^{1/2}(\tau + y)] + \frac{T_0}{\beta}\frac{y}{\sqrt{\tau}} \tag{2.9}$$

in the region LHM

$$s = \frac{T_0}{\beta}\,[A^{1/2}(\tau - y) - B^{1/2}(\tau - y)] + T_0 \tag{2.10}$$

on the bisector $\tau = y$

$$s = \frac{T_0}{\beta}\sqrt{y} - \frac{T_0}{\beta}A^{1/2}(2y) \quad (y \leqslant \beta^2), \qquad s = 0, \quad (y \geqslant \beta^2) \tag{2.11}$$

Comparing with (2.6), we obtain the jump in stresses at the shock-wave front $\tau = y$

$$[s] \equiv s(y, y+0) - s(y, y-0) = T_0$$

Figure 2 gives the graphs of the variation of stresses with time for $T_0 = 1/2$ at the sections $y = 1$ and $y = 6$. Figure 3 gives the graphs of the dependence of s on the distance from the free face of the bar ahead of and behind the shock-wave front.

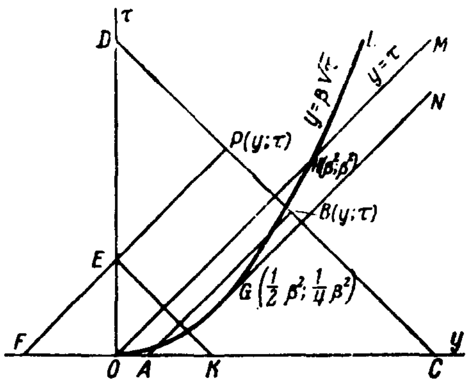

Fig. 1.

3. From these graphs it is clear that when $T_0 = 1$, the stress reaches the elastic limit at the special point O (the origin of coordinates) and on the whole section $y \geqslant \beta^2$ of the leading face of the shock-wave front, which is a consequence of the fact that the velocity of propagation of heat in the bar is finite. Moreover, this present solution differs essentially from that in [3], which was constructed on the basis of the classical linear equations of heat conduction. When $T_0 > 1$, there arise regions of plastic deformations (1), which are adjacent to the leading side of the shock-wave front and the forms of which are shown schematically in Fig. 4, where (2) is the elastic region.

In the elastic region (where $\tau \leqslant y$), the earlier solutions (2.4) and (2.5) remain valid. From these formulas and the condition $s = -1$ we find the equation of the elastic-plastic boundary

$$A^{1/2}(\tau + y) - A^{1/2}(\tau - y) = \frac{y}{\sqrt{\tau}} + \beta\frac{1 - T_0}{T_0} \qquad (y \leqslant \beta\sqrt{\tau})$$
$$y = \tau + \frac{\beta^2}{4}\frac{T_0^2 - 1}{T_0^2} \qquad (y \geqslant \beta\sqrt{\tau}) \tag{3.1}$$

We will find the value $T_0 = T_0^*$ for which the points A and B merge, i.e. starting with which all cross-sections of the bar in the course of time become plastic. Clearly, this takes place (Fig. 3) under the

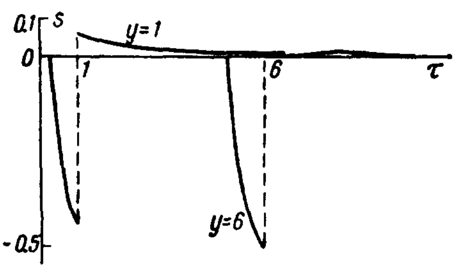

Fig. 2.

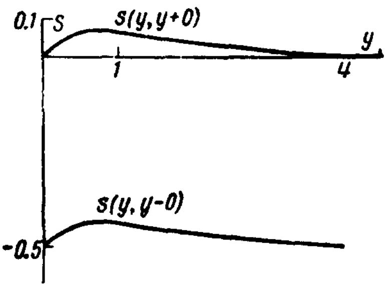

Fig. 3.

condition max $s(y,\ y - 0) = -1$. Setting the derivative ds/dy equal to zero, we obtain

$$\sqrt{y}\,(2\sqrt{{}^1/_4\beta^2 + 2y} - \beta) - \sqrt{{}^1/_4\beta^2 + 2y}\,A^{1/2}(2y) = 0, \qquad \text{or} \qquad (4y - {}^1/_2\beta^2)^2 = 0$$

Hence

$$y = \frac{1}{8}\beta^2, \qquad T_0{}^* = \frac{2\sqrt{2}}{\sqrt{6 - 4\sqrt{2}} + 2\sqrt{2} - 1} \approx \frac{7}{6} \tag{3.2}$$

In the plastic region, for a linear work-hardening material, the stresses are (Fig. 5)

$$\frac{\partial u}{\partial y} = e \equiv e^\circ + T, \qquad s = q^2 e^\circ + q^2 - 1, \qquad q^2 = \frac{E_1}{E} \tag{3.3}$$

where E_1 is the modulus of strain hardening. Whence

$$s = q^2\frac{\partial u}{\partial y} - q^2 T + q^2 - 1 \tag{3.4}$$

From (1.5) and (3.4) we obtain the equation of motion

$$\frac{\partial^2 u}{\partial \tau^2} = q^2\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial T}{\partial y}\right) \tag{3.5}$$

This equation can be solved numerically by the method of characteristics. Along the characteristics $dy = \pm\, q d\tau$ we accordingly have

$$du_\tau = \pm\, q du_y - q^2 \frac{T_0}{\beta \sqrt{\tau}}\, d\tau$$

Integrating these relations, we obtain

$$\begin{aligned} u_\tau &= \pm\, q u_y - \frac{2q^2 T_0}{\beta} \sqrt{\tau} + C_{1,2} \quad \text{when } T \neq 0 \\ u_\tau &= \pm\, q u_y - D_{1,2} + C_{1,2} \quad \text{when } T = 0 \end{aligned} \tag{3.6}$$

and then it is easy to find the stresses. The constants $D_{1,2}$ in the last formula can be found from the condition that the velocities and stresses should be continuous along the characteristics, i.e. by requiring equal values of the function $2q^2 T_0/\beta\sqrt{\tau}$ at the point of intersection of corresponding characteristics with the parabola $y = \beta\sqrt{\tau}$; the constants $C_{1,2}$ can be determined from the values of u_y and u_τ on the elastic-plastic boundary.

The solution which has been constructed for the plastic regions will be unique only for those values of q for which the characteristics $dy = \pm\, q d\tau$ intersect the elastic-plastic boundary at a single point. This limitation is a consequence of the assumption concerning linear work hardening. The calculations show that for $T_0 \geqslant 2.5$ the solution is valid for arbitrary values $0 \leqslant q < 1$.

If the solution is known in the plastic region, then the solution behind the shock-wave front can be constructed in the following manner.

Since it is known from the elastic solution that $[s] > 0$, it follows that instantaneous unloading along the line

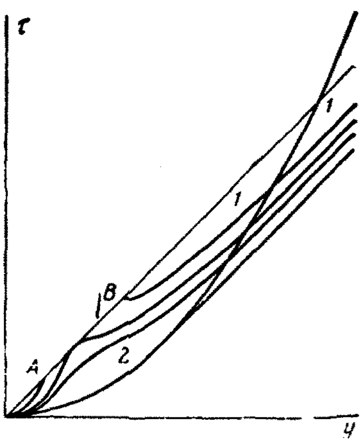

Fig. 4.

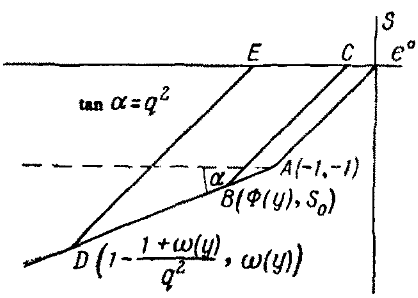

Fig. 5.

BC (Fig. 5) takes place, when passing across the shock-wave front. We introduce the functions

$$\Phi(y) = u_y(y, y-0) - T(y, y) \equiv e^\circ(y, y-0)$$
$$\Psi(y) = u_\tau(y, y-0)$$

By formula (3.3), we find (Fig. 5)

$$s_0 = q^2\Phi(y) + q^2 - 1$$

and the equation of the unloading line BC will be

$$s = e^\circ - (1 - q^2)[\Phi(y) + 1]$$

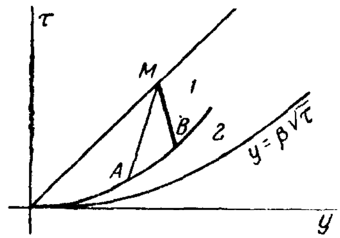

Fig. 6.

Consequently, the equation of state in the region behind the shock-wave can be written as follows

$$s = u_y - T + \alpha(y), \qquad \alpha(y) = \begin{cases} (q^2 - 1)[\Phi(y) + 1] & (\Phi(y) \leqslant -1) \\ 0 & (\Phi(y) \geqslant -1) \end{cases} \tag{3.7}$$

whereby, when $T > T_0$* for arbitrary y

$$\alpha(y) = (q^2 - 1)[\Phi(y) + 1]$$

Thus, in the region $\tau \geqslant y$ we again obtain a Cauchy problem

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \tau^2} + Q_2(y, \tau), \qquad Q_2 = T_y - \alpha'(y)$$
$$u(0, \tau) = \varphi(\tau), \qquad u_y(0, \tau) = T_0 - \alpha(0) \tag{3.8}$$

where $\varphi(\tau)$ is an as yet unknown function.

We define $\alpha_0 = \alpha(0)$. Along the elastic-plastic boundary (3.1) we have

$$s = -1, \qquad T = T_0\left[1 - \frac{1}{\beta} A^{1/2}(\tau + y) + \frac{1}{\beta} A^{1/2}(\tau - y) + \frac{1 - T_0}{T_0}\right]$$

When $y, \tau \to 0$ along (3.1)

$$A \to 0, \qquad T \to T_0\left(1 + \frac{1 - T_0}{T_0}\right) = 1$$

Consequently, $u_y = s + T$ tends to $u_0 = 0$.

Ahead of the shock-wave near the origin of coordinates (Fig. 6), by formula (3.6), we have

$$u_y(M) = \frac{u_y(A) + u_y(B)}{2} + \frac{u_\tau(A) - u_\tau(B)}{2q} + \frac{f(A) - f(B)}{2q}, \qquad f(\gamma) = \frac{2q^2 T_0}{\beta}\sqrt{\gamma}$$

When $M \to 0$, we have $u_y \to 1/2(u_0 + u_0) = 0$, and, consequently,

$\Phi(0) = -T_0$.

Similarly, it can be proved that $\Psi(0) = 0$. Thus,

$$\alpha_0 = (1 - q^2)(T_0 - 1) \tag{3.9}$$

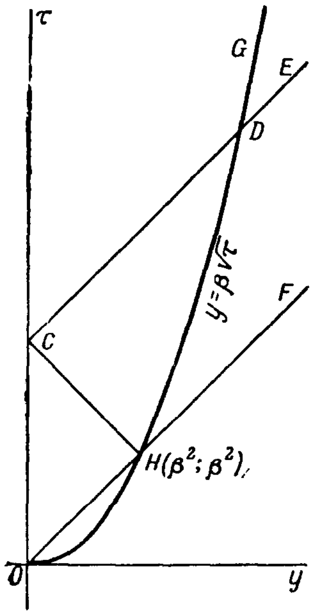

Fig. 7.

The solution of (3.8) has the form

$$u(y, \tau) = \frac{1}{2}\varphi(\tau + y) + \frac{1}{2}\varphi(\tau - y) + \frac{1}{2}\int_{\tau}^{\tau+y} T(y + \tau - \eta, \eta)\, d\eta + \frac{1}{2}\int_{\tau-y}^{\tau} T(y - \tau + \eta, \eta)\, d\eta - \int_0^y \alpha(y)\, dy \tag{3.10}$$

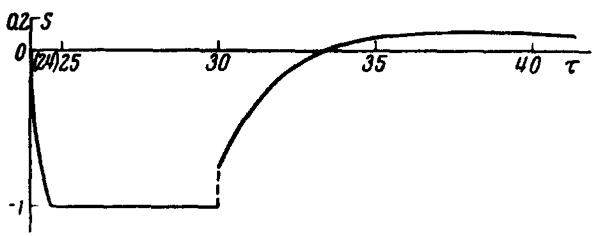

Fig. 8.

From the condition that u is continuous when $\tau = y$, we find

$$\varphi(\gamma) = -\varphi(0) + 2\int_0^{1/2\gamma} \alpha(y)\, dy - \int_0^{1/2\gamma} T(y, y)\, dy - \int_{1/2\gamma}^{\gamma} T(\gamma - y, y)\, dy + 2U(^1/_2\gamma) \tag{3.11}$$

where $U(y) = u(y, y - 0)$. From (1.9), (2.6), (3.10), (3.11), and noting that

$$dU/dy = u_y(y, y - 0) + u_\tau(y, y - 0) = \Phi(y) + \Psi(y) + T(y, y)$$

we find the stresses in the corresponding regions

$$s = F(y, \tau) - \frac{T_0}{\beta\sqrt{2}}(\sqrt{\tau + y} - \sqrt{\tau - y}) - T(y, \tau) \qquad (OHCO) \tag{3.12}$$

$$s = F(y, \tau) + \frac{T_0}{\beta\sqrt{2}}\sqrt{\tau - y} - \frac{T_0}{\beta}A^{1/2}(\tau + y) - T(y, \tau) \qquad (CHDC) \tag{3.13}$$

$$s = F(y, \tau) + \frac{T_0}{\beta\sqrt{2}}\sqrt{\tau - y} - \frac{T_0}{\beta}B^{1/2}(\tau - y) \qquad (EDHF) \tag{3.14}$$

$$s = F(y, \tau) - \frac{T_0}{\beta}[A^{1/2}(\tau + y) - A^{1/2}(\tau - y)] - T(y, \tau) \qquad (\tau CDG) \qquad (3.15)$$

$$s = F(y, \tau) - \frac{T_0}{\beta}[B^{1/2}(\tau - y) - A^{1/2}(\tau - y)] \qquad (GDE) \qquad (3.16)$$

where

$$F(y, \tau) = \frac{1}{2}\alpha\left(\frac{\tau + y}{2}\right) - \frac{1}{2}\alpha\left(\frac{\tau - y}{2}\right) + \frac{1}{2}\Phi\left(\frac{\tau + y}{2}\right) - \frac{1}{2}\Phi\left(\frac{\tau - y}{2}\right) + \qquad (3.17)$$

$$+\frac{1}{2}\Psi\left(\frac{\tau + y}{2}\right) - \frac{1}{2}\Psi\left(\frac{\tau - y}{2}\right) + \frac{1}{2}T\left(\frac{\tau + y}{2}, \frac{\tau + y}{2}\right) - \frac{1}{2}T\left(\frac{\tau - y}{2}, \frac{\tau - y}{2}\right) + T_0$$

From (3.4), (3.12) and (3.14) we find

$$[s] = \frac{1}{2}\alpha(y) + \left(\frac{1}{2} - q^2\right)\Phi(y) + \frac{1}{2}\Psi(y) + \frac{1}{2}T(y, y) - \frac{1}{2}\alpha_0 + 1 - q^2$$

In that part of the plastic region, in which the characteristics $dy = \pm\, q d\tau$ do not pass through the region $T > 0$, by (3.6), we have $u_y \equiv -1$, $u_\tau \equiv 1$. Hence, for sufficiently large y

$$\alpha(y) = (q^2 - 1)[\Phi(y) + 1] = (q^2 - 1)(u_y - T + 1) = 0$$

$$[s] = 1 - \frac{1}{2}(1 - q^2)(T_0 - 1)$$

When $T_0 > (3 - q^2)/(1 - q^2)$, the jump in the stresses on the shock-wave becomes negative and the obtained solution behind the shock-wave front is no longer valid, since the equation of state (3.7), in this case, is not valid for all values of y. The solution for $T_0 > (3 - q^2)/(1 - q^2)$ can be constructed in the following way.

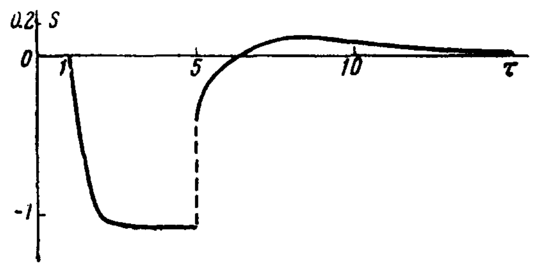

Fig. 9.

First of all we observe that, in this case, the stress exceeds the elastic limit also at the rear of the shock-wave front. Taking the solution sought to be unique, we will assume that only the rear $\tau = y + 0$ of the wave front remains plastic and that there is immediate unloading behind it. We introduce the function $\omega(y) = s(y, y = 0)$. Then the equation of the line of unloading DE (Fig. 5) will be

$$s = e^\circ + \omega(y) - \frac{\omega(y)}{q^2} + 1 - \frac{1}{q^2}$$

and the equation of state can be written down in the form

$$s = u_y - T + \Omega(y), \qquad \Omega(y) = \begin{cases} \alpha(y) & [s] \geqslant 0 \\ (q^2 - 1)/q^2 [\omega(y) + 1], & [s] \leqslant 0 \end{cases} \tag{3.18}$$

Moreover, for sufficiently small y and τ, $\Omega(y) = \alpha(y)$ always, so that $\lim s(y, y + 0) = 0$, $\lim s(y, y - 0) = -T_0$ as $y \to 0$ and $[s]_{y=0} = T_0$. Consequently, $\Omega(0) = \alpha_0$.

When $\tau \geqslant y$, the solution (3.12) to (3.17), in which $\alpha(y)$ must be replaced by $\Omega(y)$, will now be valid. The function $\omega(y)$ can be obtained from the condition $s(y, y + 0) = \omega(y)$

$$\omega(y) = \frac{q^2}{1 + q^2} [\Phi(y) + \Psi(y) + T(y, y)] - \frac{\alpha_0 q^2 - q^2 + 1}{1 + q^2} \tag{3.19}$$

It is easily verified that $s_\tau(y, y + 0) = \infty$, i.e. the assumption that there is instantaneous unloading after the passage of the wave front is fulfilled.

Figures 8 and 9 show the graphs (s, τ) at the sections $y = 5$, and $y = 30$ of the bar for $T_0 = 3$, $q = 1/2$, which graphs were obtained as the result of numerical computations.

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Translated by D.B. McV.